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RADIAL IMPACT ON AN ELASTICALLY SUPPORTED RING

By Edward Wenk, Jr., J. M. ASCE

ENGINEERING MECHANICS DIVISION

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PAPERS

RADIAL IMPACT ON AN ELASTICALLY SUPPORTED RING

By Edward Wenk, Jr.,[1] J. M. ASCE

Synopsis

The study of free vibrations of a circular ring, as developed by A. E. H. Love and S. Timoshenko, has been extended for the case of a circular ring that has radial and tangential elastic support and that is subjected to a radial impulse in the plane of the ring. Equations are developed for the frequencies of rigid-body motion and flexural vibration, and infinite series solutions are obtained for displacements and bending moments produced by an impulsive load analytically defined as a half-sine wave. In the development of the theory, the vibrations are assumed to be inextensional, and damping effects, rotatory inertia, and shear are neglected.

The theory was investigated experimentally by tests of a 24-in.-diameter ring radially supported by a large number of coil springs. Impulses of half-sine characteristic were delivered by a pendulum, and the force of impact was accurately measured with a dynamometer especially developed for this study. The parameters of elastic support and of the duration and intensity of load were varied during the test; and the strains, radial displacements, and external loads were recorded.

The observed frequencies of vibration agreed with those computed by the theory very closely up to the seventh mode of flexural motion. Also, the observed strains and displacements were found to correlate with the computed values both for the time interval during the pulse and for the subsequent free vibrations.

It is concluded that the theory is valid within the bounds of the assumptions, provided that the depth of cross section of the ring is small compared with the diameter.

Note.—Written comments are invited for publication; the last discussion should be submitted by May 1, 1953.

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Notation.—The letter symbols are defined where they first appear, in the illustrations or in the text, and are arranged alphabetically for reference in Appendix I.

Introduction

Fundamental to this investigation is the differential equation for the static bending of a thin curved element, that was established by J. Boussinesq,[2]

$$M = E I \left(\frac{1}{r + \Delta r} - \frac{1}{r} \right) = \frac{E I}{r^2} \left(\frac{\partial^2 u}{\partial \theta^2} + u \right) \ldots\ldots\ldots\ldots (1)$$

in which M is a flexural couple acting about the x-axis on a cross section of the ring—positive if it increases the curvature of the ring; E is the modulus of elasticity; I is the moment of inertia about the x-axis of a cross section of the ring; r is the mean radius of the ring; u is the radial displacement, measured positive toward the center of the ring; and θ is the central angle of the ring, measured positive counterclockwise. The same differential equation was apparently derived independently by H. Lamb.[3] These solutions have been applied to the bending analysis of a thin circular ring statically loaded in its plane.[4,5] The analysis of circular rings with radial elastic support has been studied by A. J. S. Pippard and W. E. Francis,[6] who also obtained experimental verification by static tests of wire wheels. Their instrumentation was confined to Marten optical extensometers for measuring strains in the radial spokes, from which measurements the bending moments and radial displacements of the ring were inferred. Similar theoretical analyses have been developed by L. H. Donnell and others,[7] and by M. I. Hetényi.[8]

The vibrations of a circular ring were first investigated by R. Hoppe,[9] who derived the frequency equation,

$$p^2_i = \frac{i^2(i^2 - 1)^2}{1 + i^2} \frac{g E I}{A \gamma r^4} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

in which p_i is the circular frequency of a vibrating ring at a mode designated by an integer i; g is the acceleration due to gravity; A is the area of a cross section of the ring; and γ is the density of the material. Mr. Love[10] later studied the

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[2] "Résistance d'un anneau à la flexion, quand sa surface extérieure supporte une pression normal, contante par unité de longuer de sa fibre moyenne," by J. Boussinesq, *Comptes Rendus*, Paris, France, Vol. 97, 1883, p. 843.

[3] "On the Flexure and Vibrations of a Curved Bar," by H. Lamb, *Proceedings*, London Math. Soc., Vol. 19, 1888, p. 365.

[4] "Strength of Materials," by S. Timoshenko, D. Van Nostrand, Inc., New York, N. Y., Vol. 2, 2d Ed., 1940, p. 101.

[5] *Ibid.*, p. 218.

[6] "On a Theoretical and Experimental Investigation of the Stresses in a Radially Spoked Wire Wheel under Loads Applied at the Rim," by A. J. S. Pippard and W. E. Francis, *Philosophical Magazine*, Series 7, Vol. 11, No. 69, 1931, pp. 233–285.

[7] "Analysis of Spoked Rings," by L. H. Donnell, H. B. Gibbons, and E. L. Shaw, *Journal of Applied Mechanics*, Vol. 8, No. 2, 1941, pp. 67–73.

[8] "Beams on Elastic Foundations," by M. I. Hetényi, Univ. of Michigan Press, Ann Arbor, Mich., 1946.

[9] "Vibrationen eines Ringes in seiner Ebene," by R. Hoppe, *Journal für die Reine und Angewandt Mathematik*, Vol. 73, 1871, pp. 158–170.

[10] "A Treatise on the Mathematical Theory of Elasticity," by A. E. H. Love, Dover Publications, 4th Ed., 1944, pp. 451–453.

fundamental equations of motion for both a complete ring and an incomplete ring with a cross section that was also circular. For these solutions, the important assumption was made that the center line of the ring is free of any extension.

The flexural vibration of a circular ring, when the possible extension of the center line is considered, is discussed by F. W. Waltking,[11] and by K. Federhofer.[12] Vibrations of ring-shaped frames out of the plane of the frame are discussed by J. H. Michell,[13] by Mr. Love,[10] and by F. H. Brown,[14] and torsional vibrations are also treated by Mr. Love.[10]

More recently, G. F. Carrier[15] has given a mathematical analysis of vibrations of a circular ring that is rotating at high speed so as to develop important centrifugal effects and that is either rigidly or elastically supported by radial spokes.

As a further development of the mechanics of circular rings, there is investigated in this paper the dynamical behavior of a ring with both radial and tangential elastic support, subjected to a concentrated radial force that has the time characteristic of a half-sine wave. A frequency equation for flexural and rigid-body motion is derived and, by application of procedures used by Mr. Timoshenko[16] and W. H. Hoppmann, II,[17] for the impact loading of straight beams, normal mode solutions are obtained for bending displacements in the plane of the ring. These solutions developed in 1948 may be considered only as first-order approximations; a second-order approximation that leads to the buckling load for an elastically supported ring has also been developed and is contained in a Taylor Model Basin (TMB) report.

The validity of the theoretical analysis was investigated experimentally by tests of a circular ring with a diameter of 24 in., radially supported by thirty-eight coil springs intended to simulate pure radial elastic support. The ring was subjected to a radial impulse of controlled duration and intensity during which time the impulse and the resulting strains and displacements were measured with electrical instruments. The ring was also subjected to a harmonically varying force over a continuous frequency spectrum to determine the frequencies of flexural vibration by observation of resonance.

Several new experimental instruments and techniques were developed for these tests, including a method for applying, controlling, and precisely measuring a transient impulse. These are described in detail.

The paper concludes with a comparison of theoretical and experimental results for the various cases of loading.

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[11] "Schwingungszahlen und Schwingungsformen von Kreisbogentragern," by F. W. Waltking, *Ingenieur-Archive*, Vol. 5, 1934, pp. 429–449.

[12] "Über die Eigenschwingungen des senkrecht zu einer Ebene schwingenden Kreisbogens," by K. Federhofer, Akadamie der Wissenschaften, Wien, Berlin, Germany, ab IIa, Vol. 145, 1936, pp. 29–50.

[13] "The Small Deformations of Curves and Surfaces with Application to the Vibration of a Helix and Circular Rings," by J. H. Michell, *Messenger of Mathematics*, Vol. 19, 1890, pp. 68–82.

[14] "Lateral Vibrations of Ring-Shaped Frames," by F. H. Brown, *Journal*, Franklin Inst., Vol. 218, No. 1, 1934, pp. 41–48.

[15] "On the Vibrations of the Rotating Ring," by G. F. Carrier, *Quarterly of Applied Mathematics*, Vol. 3, No. 3, 1945, pp. 235–245.

[16] "Vibration Problems in Engineering," by S. Timoshenko, D. Van Nostrand, Inc., New York, N. Y., 2d Ed., 1937, pp. 405–410.

[17] "Impact of a Mass on a Damped Elastically Supported Beam," by W. H. Hoppmann, II, *Journal of Applied Mechanics*, Vol. 15, No. 2, June 1948, pp. 125–136.

Theoretical Analysis

The subject of this investigation is a plane circular ring of constant and symmetrical cross section whose dimensions are small compared with the radius of the center line; one of the principal axes of the cross section is assumed to be situated in the plane of the ring. Furthermore, the ring is considered to have uniform radial and tangential supports that are completely elastic and that exercise constraints at the centroid of the cross section of the ring. Damping and body forces are neglected in the analysis, and external forces are restricted to radial orientation in the plane of the ring.

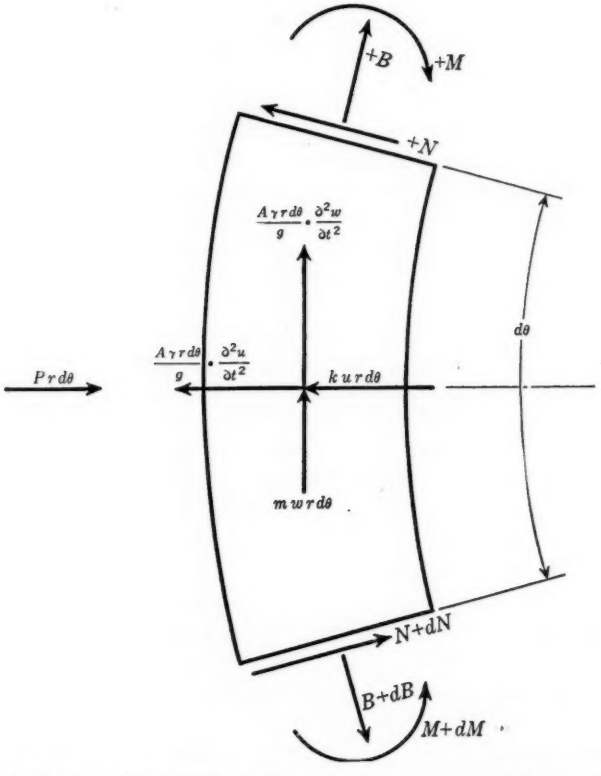


Fig. 1.—Equilibrium Diagram of an Element of an Elastically Supported Ring

The Differential Equation of Motion.—Consider first an element of the ring as shown in Fig. 1. Equations of equilibrium are derived, taking into account the internal shear force denoted by N; the normal force denoted by B; the flexural moments, M (all imposed by contiguous ring elements); the radial and tangential components of particle accelerations; the external loads, P; and the radial and tangential elastic supports.

In the radial direction, the forces are as follows:

$$\frac{\partial N}{\partial \theta} + B = \frac{A\,\gamma\,r}{g}\frac{\partial^2 u}{\partial t^2} + k\,u\,r - P\,r \ldots\ldots\ldots\ldots\ldots (3a)$$

in which k is the modulus of elastic support expressed in units of pounds per inch of circumference per inch of radial displacement, u designates radial displacement taken positive toward the center, and t is the time variable.

In the tangential direction, the forces are

$$\frac{\partial B}{\partial \theta} - N = \frac{A \gamma r}{g} \frac{\partial^2 w}{\partial t^2} + m w r \quad \ldots\ldots\ldots\ldots\ldots\ldots (3b)$$

in which m is the modulus of tangential support and w is the tangential displacement taken positive in the counterclockwise direction.

If rotatory inertia is neglected, the equation of equilibrium involving moments is

$$\frac{\partial M}{\partial \theta} + N r = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3c)$$

Terms corresponding to effects of shearing distortion have also been neglected, but the possible error thus introduced has been found by Mr. Timoshenko[18] to be unimportant for bars with vibrational wave lengths of at least ten times the depth of the cross section.

The tangential displacements, w, are now assumed to occur in such a form that the extension of the center line is zero. This condition of inextensional vibration was expressed by Mr. Love as

$$u = \frac{\partial w}{\partial \theta} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

If the change in curvature in terms of bending displacements is represented by the equation given by Mr. Boussinesq[2] for bending of a curved beam—

$$M = E I \left(\frac{1}{r + \Delta r} - \frac{1}{r} \right) = \frac{E I}{r^2} = \left(\frac{\partial^2 u}{\partial \theta^2} + u \right) \quad \ldots\ldots\ldots\ldots (5)$$

—then, by use of Eqs. 4 and 5, the equations of equilibrium (Eqs. 3a, 3b, and 3c) may be reduced to eliminate the terms involving M, N, and B. This gives the partial differential equation of motion in terms of tangential displacements:

$$\frac{E I}{r^4} \left(\frac{\partial^6 w}{\partial \theta^6} + 2 \frac{\partial^4 w}{\partial \theta^4} \frac{\partial^2 w}{\partial \theta^2} \right) + k \frac{\partial^2 w}{\partial \theta^2} - m w - \frac{\partial P}{\partial \theta} = \frac{A \gamma}{g} \frac{\partial^2}{\partial t^2} \left(w - \frac{\partial^2 w}{\partial \theta^2} \right) \ldots (6a)$$

By differentiating Eq. 6a with respect to θ and applying Eq. 4, a relationship may be obtained for the radial displacements:

$$\frac{E I}{r^4} \left(\frac{\partial^6 u}{\partial \theta^6} + 2 \frac{\partial^4 u}{\partial \theta^4} + \frac{\partial^2 u}{\partial \theta^2} \right) + k \frac{\partial^2 u}{\partial \theta^2} - m u - \frac{\partial^2 P}{\partial \theta^2} = \frac{A \gamma}{g} \frac{\partial^2}{\partial t^2} \left(u - \frac{\partial^2 u}{\partial \theta^2} \right) \ldots (6b)$$

In the usual fashion, it is assumed that the radial displacement expressed by u is the product of a characteristic function, $U(\theta)$, and a time function, $\Gamma(t)$; thus,

$$u = U \Gamma \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

[18] "Vibration Problems in Engineering," by S. Timoshenko, D. Van Nostrand, Inc., New York, N. Y., 2d Ed., 1937, p. 337.

With the variables thus separated, the time function represented by Γ is found (by substitution of Eq. 7 in Eq. 6*b*) to be in the simple harmonic form, of circular frequency p, as follows:

$$\Gamma = C \cos p\,t + D \sin p\,t = \cos (p\,t + \delta) \ldots\ldots\ldots\ldots (8)$$

in which C, D, and δ are constants. This condition yields the following ordinary differential equation for U, the characteristic function of radial displacement:

$$\frac{d^6U}{d\theta^6} + 2\frac{d^4U}{d\theta^4} + \left[1 + \frac{\left(k - \frac{A\,\gamma}{g}p^2\right)}{E\,I} r^4\right]\frac{d^2U}{d\theta^2} + \frac{r^4}{E\,I}\left(\frac{A\,\gamma\,p^2}{g} - m\right)U = \frac{1}{\Gamma}\left(\frac{r^4}{E\,I}\right)\frac{d^2P}{d\theta^2} \ldots (9a)$$

The homogeneus equation describing the motion of the freely vibrating ring, when $P = 0$, is of the form:

$$\frac{d^6U}{d\theta^6} + 2\frac{d^4U}{d\theta^4} + \nu\frac{d^2U}{d\theta^2} + \mu\,U = 0 \ldots\ldots\ldots\ldots\ldots (9b)$$

in which the coefficients appearing in Eq. 9*a* have been replaced by the symbols ν and μ. The solution of this equation is given by Mr. Love[10] to be

$$U = \sum_{n=1}^{n=3} (C_n \cos i_n\,\theta + D_n \sin i_n\,\theta) \ldots\ldots\ldots\ldots\ldots (10)$$

in which n is an integer and i_1, i_2, and i_3 are roots of the equation,

$$i^6{}_n - 2\,i^4{}_n + \nu\,i^2{}_n - \mu = 0 \ldots\ldots\ldots\ldots\ldots (11)$$

Eq. 11 follows from the substitution of Eq. 10 in the differential equation (Eq. 9*b*).

Frequency Equation for Free Vibration.—In the usual dynamical analysis of bars, physical boundary conditions would be available from which the constants of integration of the differential equation could be determined. In the problem of a complete ring, however, no boundary conditions exist for bending displacements or their functions. However, the requirement exists that the solution be univalued in θ. Thus, for a complete ring, there is the recurrence condition that

$$u_\theta = u_{\theta+2\pi} = u_{\theta+4\pi} = u_{\theta+2n\pi} \ldots\ldots\ldots\ldots\ldots (12)$$

in which $n = 0, 1, 2, 3, 4, 5, \ldots \infty$. This requires that the i_n-terms appearing in Eq. 10 must be integers, which condition establishes the frequency equation. By employing the explicit coefficients of Eq. 9*a* ,Eq. 11 may be solved for the frequency p_i as

$$p^2{}_i = \frac{g}{A\,\gamma}\frac{1}{1 + i^2}\left[\frac{E\,I}{r^4}(1 - i^2)^2\,i^2 + k\,i^2 + m\right] \ldots\ldots\ldots (13)$$

Here, the integer values of i_n are represented simply by i, a quantity greater than unity which defines the number of wave lengths of flexural vibration that

may develop in the circumference. The cases wherein i equals zero and unity are the cases of rigid-body rotation and translation of the ring, respectively.

Radial Displacements and Bending Moments Produced by an Harmonically Varying Radial Force.—From Eqs. 8 and 10, the radial displacements may be expressed as an infinite series, as noted previously; thus:

$$u = \sum_{n=1}^{n=\infty} (C_n \cos i_n \theta + D_n \sin i_n \theta) \cos (p t + \delta) \ldots\ldots\ldots (14)$$

A single concentrated external force applied at $\theta = 0$ produces symmetrical vibrations, and consequently the odd functions $\sin i_n \theta$ must not appear in the displacement function. This is accomplished by making the coefficients D_n equal to zero. Then,

$$u = \sum_{i=1}^{\infty} a_i \cos i \theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots (15a)$$

and

$$w = \sum_{i=1}^{\infty} \frac{a_i}{i} \sin i \theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots (15b)$$

in both of which a_i-values are functions of time and are taken as the generalized coordinates.

The Lagrange equation of motion may be restated as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{a}_i}\right) + \frac{\partial V}{\partial a_i} = Q_i \ldots\ldots\ldots\ldots\ldots\ldots\ldots (16)$$

in which Q_i is the external generalized force, measured positive toward the center; the dotted quantities in this and subsequent equations are derivatives taken with respect to time t. The potential energy, V, of the system is expressed by

$$V = \frac{1}{2}\int \frac{M^2}{E\,I}\,ds + \frac{k}{2}\int u^2\,ds + \frac{m}{2}\int w^2\,ds$$

$$= \frac{E\,I}{2\,r^4}\int_0^{2\pi} \left(\frac{\partial^2 u}{\partial \theta^2} + u\right)^2 r\,d\theta + \frac{k}{2}\int_0^{2\pi} u^2\,r\,d\theta + \frac{m}{2}\int_0^{2\pi} w^2\,r\,d\theta \ldots (17a)$$

in which ds is an increment of the circumference.

Similarly, the kinetic energy, T, of the vibrating ring is

$$T = \frac{A\,\gamma}{2\,g}\int_0^{2\pi} (\dot{u}^2 + \dot{w}^2)\,r\,d\theta \ldots\ldots\ldots\ldots\ldots\ldots (17b)$$

By substitution of the series solutions (Eq. 15a), the Lagrange equation (Eq. 16) then yields:

$$\frac{\pi\,r\,A\,\gamma}{g}\left(1 + \frac{1}{i^2}\right)\ddot{a}_i + \left[(1 - i^2)^2\,\frac{\pi\,E\,I}{r^3} + \pi\,k\,r + \frac{\pi\,m\,r}{i^2}\right] a_i = Q_i \ldots (18a)$$

This equation reduces to—

$$\ddot{a}_i + p^2_i\, a_i = \frac{g}{\pi\, r\, A\, \gamma}\left(\frac{i^2}{1+i^2}\right) Q_i \ldots\ldots\ldots\ldots\ldots (18b)$$

—the solution of which is

$$a_i = H_i \cos p_i\, t + F_i \sin p_i\, t + \frac{1}{p_i}\frac{g}{\pi\, r\, A\, \gamma}\frac{i^2}{1+i^2}\int_0^{t_1} Q_i \sin p_i\,(t_1 - t)\, dt \ldots (19)$$

in which t_1 is a particular time at which the behavior is described. The constants of integration, H_i and F_i, depend on initial conditions of displacement and velocity of the ring and are taken here to be zero.

To study the case of steady-state forced vibration, a radial force P, concentrated at $\theta = 0$, is applied with simple harmonic oscillation of frequency ω and maximum intensity P_o so that $P = P_o \sin \omega\, t$. On a generalized coordinate:

$$\text{Increment of work} = P\, \Delta u = P\, \Delta a_i \cos i\, \theta = Q_i\, \Delta a_i \ldots\ldots\ldots (20)$$

The generalized force thus becomes

$$Q_i = P \cos i\, \theta = P_o \sin \omega\, t \cos i\, \theta \,|_{\theta=0} = P_o \sin \omega\, t \ldots\ldots\ldots (21)$$

Denoting the coefficients by $\alpha = \frac{g}{\pi\, r\, A\, \gamma}$, $\beta = \frac{g\, E\, I}{\pi\, r^3\, A\, \gamma}$, $i_a = \frac{i^2}{1+i^2}$, and $i_b = \frac{i^2\,(1-i^2)}{1+i^2}$, and integrating the particular integral of Eq. 19 corresponding to the transient response, the generalized coordinates or amplitude functions corresponding to the steady-state forced vibration are given by

$$a_i = \frac{P_o\, \alpha\, i_a}{P_i\,(\omega^2 - p^2_i)}\,(\omega \sin p_i\, t_1 - p_i \sin \omega\, t_1) \ldots\ldots\ldots\ldots (22)$$

If η_i is equal to the ratio $\frac{\omega}{p_i}$ of forcing frequency to the frequency corresponding to any mode of response, then (by Eq. 15*a*) the radial displacements are given by

$$u_1 = P_o\, \alpha \sum_{i=1}^{\infty} \frac{i_a}{p^2_i\,(\eta^2_i - 1)}\,(\eta_i \sin p_i\, t - \sin \omega\, t) \cos i\, \theta \ldots\ldots\ldots (23a)$$

Similarly, the equation for bending moment is found from Eq. 5 to be

$$M_1 = P_o\, \beta \sum_{i=1}^{\infty} \frac{i_b}{p^2_i\,(\eta^2_i - 1)}\,(\eta_i \sin P_i\, t - \sin \omega\, t) \cos i\, \theta \ldots\ldots\ldots (23b)$$

The time variable, t_1, has been changed to t for convenience in notation. The significance of the subscript numeral for u and M is discussed subsequently.

Thus Eqs. 23 define the response of the elastically supported ring to a steady-state forced vibration. The frequencies of free vibration are given by Eq. 13.

Radial Displacements and Bending Moments Produced by Transient Loading.—Consider the response of the system to a radial impulse defined as a half-sine

wave of duration τ and a maximum intensity P_o (see Fig. 2). Following the procedure adopted by Mr. Hoppmann[17] for the case of an infinite bar,—a harmonic force, P' of identical amplitude and frequency is superposed on P, as shown in Fig. 2. The P' force is initiated 180° out of phase with P, that is, at a time t equal to τ which is equal to π/ω.

The time variable for the force P' is denoted by t' and the following relationships obtain between the time coordinates t and t':

$$t' = t - \tau \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (24a)$$

and

$$t'_1 = t_1 - \tau \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (24b)$$

in which t_1 is the instant at which response is to be determined.

The response of the system to the force P' may be treated by transforming the time variable, so that an expression is obtained for the generalized coordi-

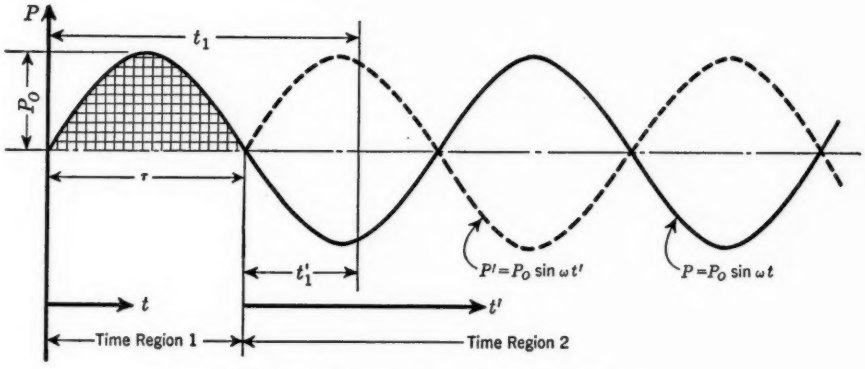


FIG. 2.—DEVELOPMENT OF A HALF-SINE PULSE

nate a'_i similar to that for a_i given by Eq. 22.

$$a'_i = \frac{P_o \alpha i_a}{p_i (\omega^2 - p^2_i)} \left[\omega \sin p_i (t_1 - \tau) - p_i \sin \omega (t_1 - \tau) \right] \ldots\ldots (25)$$

The behavior of the system after the transient disturbance ceases is obtained by superposing the steady-state forced vibration produced by P' upon that produced by P so that

$$a''_i = a_i + a'_i \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (26)$$

in which a''_i represents the new set of generalized coordinates corresponding to the transient force $P - P'$. By making use of Eqs. 22 and 25 and noting that $\cos \omega \tau = -1$ and $\sin \omega \tau = 0$, Eq. 27 results—

$$a''_i = \frac{P_o \alpha i_a \omega}{P_i (\omega^2 - p^2_i)} [(1 + \cos p_i \tau) \sin p_i t_1 - \sin p_i \tau \cos p_i t_1] \ldots (27)$$

The expressions corresponding to a''_i for radial displacement and bending moment, after convenient trigonometric simplification, reduce to

$$u_2 = P_o \alpha \sum_{i=1}^{\infty} \frac{2 \eta_i \cos \frac{p_i \eta}{2}}{p^2_i (\eta^2_i - 1)} i_a \cos i\theta \sin p_i \left(t - \frac{\tau}{2}\right) \ldots\ldots\ldots (28a)$$

and

$$M_2 = P_o \beta \sum_{i=1}^{\infty} \frac{2 \eta_i \cos \frac{p_i \tau}{2}}{p^2_i (\eta^2_i - 1)} i_b \cos i\theta \sin p_i \left(t - \frac{\tau}{2}\right) \ldots\ldots\ldots (28b)$$

Again, the time variable has been changed from t_1 to t for convenience in notation.

The response of the ring to a transient pulse is computed for $t < \tau$ by Eqs. 23 and for $t > \tau$ by Eqs. 28. The symbols for displacements (u) and bending moments (M) for the two time regions corresponding to $t < \tau$ and $t > \tau$ have been given the subscripts 1 and 2, respectively.

Static Loading Treated As a Limiting Case of Transient Loading.—Implicit in this dynamic analysis is the solution for the case of static loading. By consideration of the limiting case where $\omega \rightarrow 0$ and $\omega t \rightarrow \frac{\pi}{2}$, the radial displacements and bending moments as given by Eqs. 23 reduce to

$$u_s = P_o \alpha \sum_{i=1}^{\infty} \frac{i_a}{p^2_i} \cos i\theta \ldots\ldots\ldots\ldots\ldots\ldots (29a)$$

and

$$M_s = P_o \beta \sum_{i=1}^{\infty} \frac{i_b}{p^2_i} \cos i\theta \ldots\ldots\ldots\ldots\ldots\ldots (29b)$$

in which the subscript s denotes the behavior under static load. The transient response after the disturbance as given by Eqs. 28 is, of course, zero.

The Solution in Series Form.—Eqs. 23 define bending moments and radial displacements during forced harmonic vibration and during the interval of action of a transient half-sine pulse, and are in recognizable normal mode form.

It is immediately apparent from these equations that resonance will occur at any mode in which $\eta_i = 1$—that is, when $\omega = p_i$. If such unique cases are excluded, convergence can be proved by examining separately the various factors that appear in each series term, and by application of the ratio test of succeeding terms. Convergence is proved for the bending moments defined by Eq. 23*b*, except that the presence of the $(1 - i^2)$-term in the numerator of each series term retards convergence.

A similar test of Eqs. 28 (for free vibration after termination of a transient) also discloses convergence, even when $\eta_i = 1$.

Comparison of the Series Solution with a Closed-Form Solution for the Case of Static Load.—The differential equation of the elastically supported ring will very easily yield a closed-form solution with a static load concentrated at

$\theta = 0$, thus providing an opportunity to check both the accuracy of the derivation and the rate of convergence of the functions. With available closed-form solutions,[8] and the series of solutions of Eqs. 29, computations were made of the bending moments and radial displacements in a ring statically loaded at $\theta = 0$; the dimensions corresponded to the ring employed in later tests. The series solution was extended through ten terms; the numerical results of both calculations are given in Table 1. The locations of the stations are shown subsequently in Fig. 5.

Experimental Analysis of a Radially Supported Ring

Because of the simplifying assumptions necessary to render feasible a mathematical solution, a collateral experiment was performed to confirm the analysis. For this purpose, measurements were required of (1) the frequencies of various modes of flexural vibration, and (2) the response of the ring to a transient pulse of half-sine characteristic.

TABLE 1.—Comparison of Series Solution with Closed-Form Solution for Static Behavior of Radially Supported Ring

Orientation		Radial Displacement, u, in Mils		Bending Moment, M, in Inch-Pounds	
Station	Degrees θ	Series solution	Closed-form solution	Series solution	Closed-form solution
1	0	8.911	8.98	−1.340	−1.70
A	4.7	8.745	...	−1.237	−1.24
B	23.7	5.687	...	0.060	0.002
D	61.6	−0.962	...	0.457	0.413
E	80.5	−2.316	...	0.218	0.290
11	94.7	−2.655	−2.43	...	...
J	175.3	−1.847	...	−0.120	−0.110
21	189.4	−1.863	−1.84	...	...

Ring parameters: $k = 6.8$, $m = 0$, $E = 29.2$, $\gamma = 0.292$, $P = 1$ at $\theta = 0$, $r = 12$ in., cross section 1 in. by $\frac{1}{8}$ in.

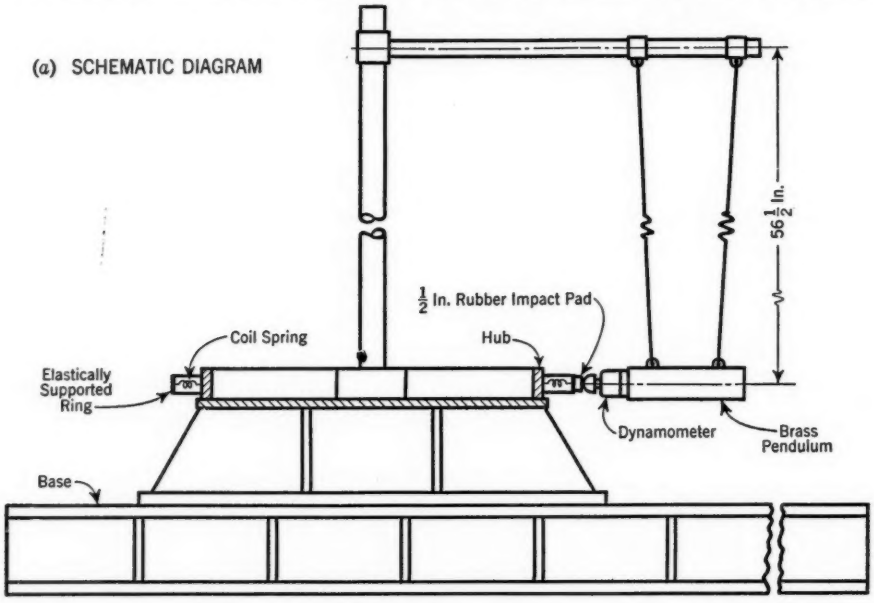
For a ring of given dimensions, the transient reponse is a function of duration and intensity of the pulse as well as of the elastic support of the ring, and therefore it was necessary to provide means for developing a half-sine impulse and for varying its duration and intensity.

The pulse generator used to deliver to the ring a transient pulse of controllable intensity and duration, and the instrument for measuring the applied load are briefly described in this section; the design and operation of all test apparatus have been discussed in considerable detail elsewhere.[19]

Description of the Ring and Elastic Support.—The ring selected for the test had a diameter of 24 in. and a rectangular cross section of 1 in. by 0.125 in; it was manufactured from flat bar stock by rolling a hoop and welding the two ends together. The material was cold-rolled steel with a modulus of elasticity of 29.2×10^6 lb per sq in.

[19] "A Theoretical and Experimental Investigation of a Dynamically Loaded Ring with Radial Elastic Support," by E. Wenk, Jr., *TMB Report No. 704*, David Taylor Model Basin, Washington, D. C., July, 1950.

For test purposes, only radial support was desired. Two different stiffnesses of elastic support for the ring were selected—one to increase the frequency of the first flexural mode of vibration of the unconstrained ring by a multiple of



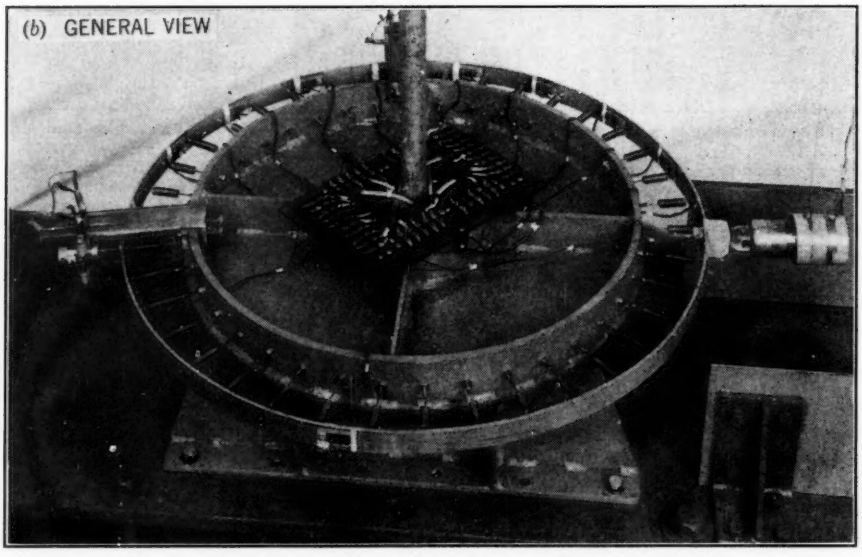


FIG. 3.—APPARATUS FOR DIRECT LOADING OF THE RING BY A PENDULUM

two, and the other to increase it by a multiple of four. The corresponding required values of k were computed to be 6.33 lb per in. per in. and 31.6 lb per in. per in., respectively. After a study of various systems to furnish elastic sup-

port, coil springs were selected, oriented radially, and spaced at 2-in. intervals around the periphery. As shown in Fig. 3, the ring was positioned horizontally, and its weight was supported by the lateral stiffness of the springs. To provide elastic support with radial displacements in both directions, the springs were installed with uniform initial tension.

All coil springs were individually calibrated to check stiffness prior to installation, and the values of k actually obtained were 6.80 lb per in. per in. and 31.6 lb per in. per in. plus or minus 5%; these values have been employed in all calculations.

Excitation of Steady-State Forced Vibrations.—For the purpose of determining natural frequencies of flexural vibration before impact tests, a heavy-duty loud speaker element, driven by an audio-oscillator, was attached to the ring for excitation over a continuous frequency spectrum. Resonance was very easily detected by sounds emanating from the vibrating ring, as well as by visual observation of strain-gage signals; driving power was sufficient to excite the eighth flexural mode without difficulty.

Pulse Generator for Applying Transient Loads.—To deliver a radial impulse with a pure half-sine characteristic, frequent use has been made of a small sphere dropped onto the test specimen from a controlled height. Computations of the duration, intensity, and shape of the applied load were made by the application of the contract theory of H. Hertz.[20] This indirect estimate of the impact characteristic may introduce undesirable errors. Furthermore, a limit to the validity of the contact theory occurs when the specimen being struck has diminutive mass or stiffness compared with that of the striking sphere. Also implicit in the Hertz theory are short durations of contact compared with the fundamental period of ring vibration. Because longer durations of contact and greater accuracy were desired for this investigation, it was necessary to explore other methods of dynamically loading the ring.

From preliminary experiments, it was found that the desired half-sine pulse could be developed readily by striking the ring with a small body moving at constant velocity. Furthermore, the critical parameters—intensity and duration of impact—could be independently controlled as a function of the mass and the velocity of the moving impact body, and of the mass and the stiffness of the elastically supported ring. Approximate relationships defining the mechanics of impact were derived for use in the design of the apparatus and for control during the test.

These were as follows:

$$\tau = \pi \sqrt{\frac{u_s/P_o\,(W_r + W_p)}{g}} \qquad (30a)$$

$$P_o = \frac{u_{s\max}}{u_s/P_o} = \frac{2\,W_p \sqrt{W_p + W_r}}{(2\,W_p + W_r)\sqrt{Y\,g}}\,v \qquad (30b)$$

and

$$v = \sqrt{\frac{g}{L}}\,d \qquad (30c)$$

[20] "Über die Berechnung Fester Elastichen Köper," by H. Hertz, *Journal für die Reine und Angewandt Mathematik*, Vol. 92, 1881, pp. 155–170.

in which P_o is the maximum intensity of the pulse; W_r is the combined weight of the ring and coil spring system; W_p is the weight of the impact body; v is the terminal velocity of the impact body, at the time of impact; u_s is the static deflection of the ring at the point of application; L is the length of the pendulum; and d is the stroke of the pendulum.

TABLE 2.—Comparison Between Observed and Computed Characteristics of Transient Loading

Modulus of elastic foundation, k (lb per in. per in.)	Computed unit static deflection, $\frac{u_s}{P_o}$ (in. per lb)	Weight of pendulum W_p (lb)	Impact pad during test	Stroke of pendulum, d (in.)	Duration τ (Sec)		Loading, P_o (Lb)	
					Computed	Observed	Computed	Observed
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
6.8	0.0090	5.2	yes	...	0.043	0.047	...	...
6.8	0.0090	2.2	yes	6	0.034	0.032	6.5	7.3
31.6	0.0021	2.2	no	...	0.016	0.016	...	...
31.6	0.0021	2.2	yes	18	0.016	0.023	41.3	39.7
31.6	0.0021	1.2 (ball)[a]	no	...	0.014	0.012	...	...

[a] In some tests a ball 2 in. in diameter was substituted for the brass impact body to confirm the loading relationships and to verify inapplicability of the Hertz contact theory in this experiment.

A comparison between observed characteristics and those computed by Eqs. 30, given in Table 2, shows agreement, within 10% for the duration values and for the intensity values.

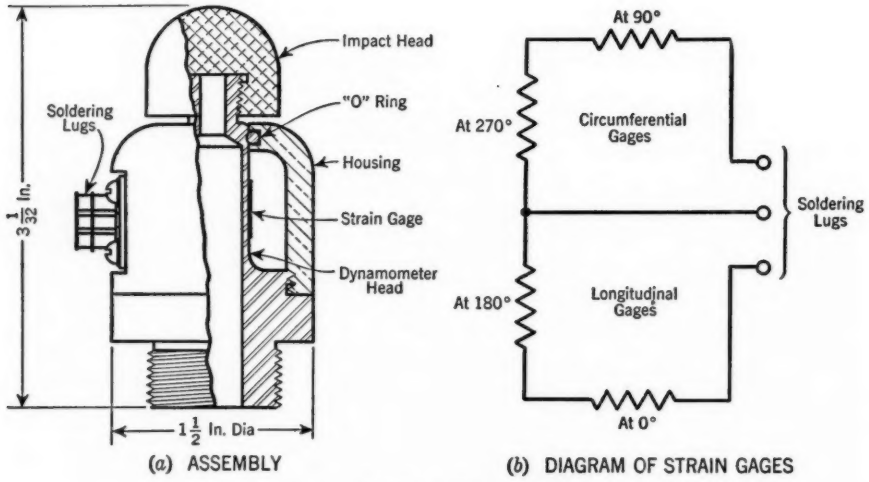


Fig. 4.—The Load Dynamometer

For the tests of the ring, the impact body was designed as a pendulum composed of a cylindrical brass bar having a 2-in. diameter and a length sufficient to provide the mass required for a specified loading characteristic. The bar was suspended on phosphor-bronze wires as shown in Fig. 3.

The Load Dynamometer.—To measure the impact load, a special dynamometer was developed that permitted remote electrical recording of the load simultaneously with measurements of ring response. This dynamometer was designed as a short, cylindrical steel tube, $\frac{1}{2}$ in. in diameter, fitted with electrical strain gages oriented for maximum response to an axially applied load and minimum response to bending and torsion, as shown in Fig. 4. Steel for the dynamometer was SAE 4340, and the housing was of aluminum. Lucite, aluminum, and steel were all satisfactory materials for impact heads. Complete details are available in a TMB report.[19] The dynamometer was cali-

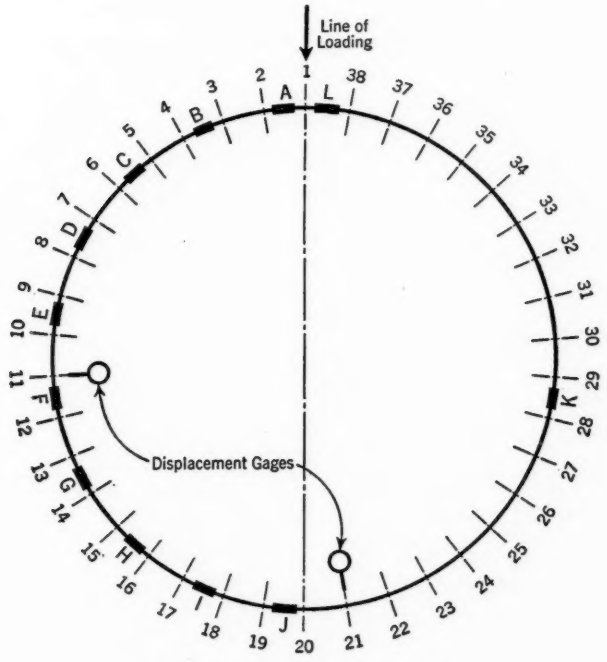


FIG. 5.—ARRANGEMENT OF GAGES

brated statically, and its response was linear and completely free of temperature instability. Its sensitivity, which depended on wall thickness was 2.37 micro-in. per in. per lb of load, permitting the measurement of forces as small as 5 lb. Its natural frequency which depended on the length was high enough to insure accurate detection of forces having durations as short as 0.01 milli-sec. Type A-B wire resistance strain gages were used in the dynomometer.

For these tests, a TMB type 1A strain indicator[21] was used with the dynamometer because of its great sensitivity. It operates on a carrier frequency of 2,200 cycles per sec and has a linear response from 0 cycles per sec to 200 cycles per sec, which was ample for these tests, in which the duration of load always exceeds 15 milli-sec.

[21] "A Carrier-Type Strain Indicator," by G. W. Cook, *TMB Report No. 565*, David Taylor Model Basin, Washington, D. C., November, 1946.

Instrumentation to Measure Strains and Displacements.—The response of the ring was considered definable in terms of bending moments and radial displacements, and instrumentation was provided for their measurement.

The bending moments were computed from strains measured by wire-resistance gages, mounted in pairs on both faces of the ring at each station so as to double the electrical output and insure temperature compensation. Gage locations are shown in Fig. 5, in which the numbers indicate spring locations and the letters indicate strain gages. Signals from the gages were carried through TMB type 1A amplifier equipment which included a calibration system.

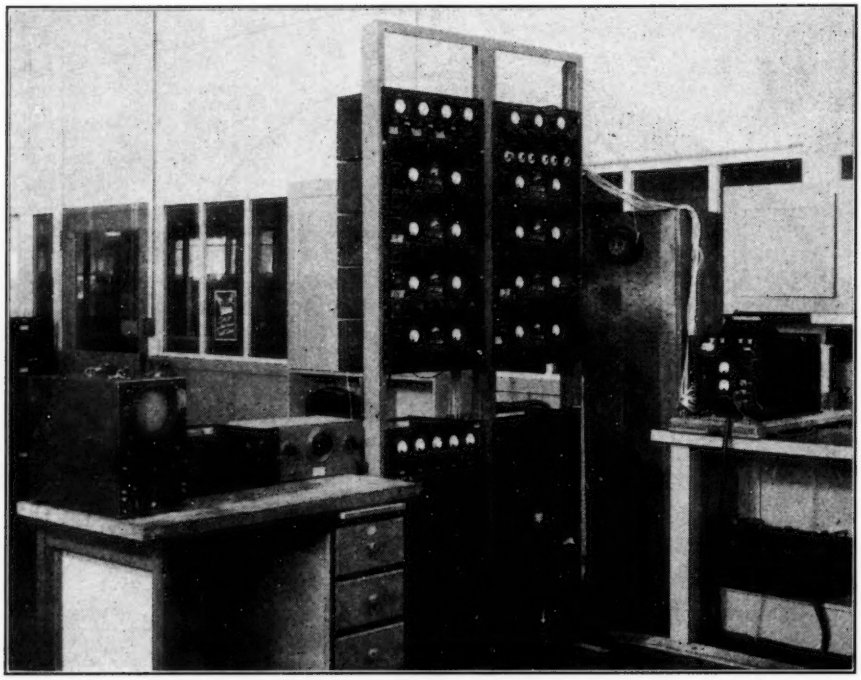


FIG. 6.—AUXILIARY ELECTRONIC AND RECORDING EQUIPMENT

Radial displacements were measured with linear variable differential transformer gages.[22] These gates were used with an auxiliary driving unit[23] that furnished 1,000-cycle alternating current to the gage and included sense discriminating and rectifier circuits that give a direct-current output (with a sense of direction) proportional to the measured displacement. The gages were calibrated after they were in place by means of a mechanical device with a dial indicator.

[22] "The Linear Variable Differential Transformer," by H. Schaevitz, *Proceedings*, Soc. for Experimental Stress Analysis, Vol. 4, No. 2, 1947, pp. 79–88.

[23] "Differential Transformer Unit Driver and Demodulator Type 10L," *TMB Electronic Instrument Instruction Manual*, David Taylor Model Basin, Washington, D. C., March, 1949.

A string oscillograph was used for recording amplified electrical signals from the various pickups. A 60-cycle-per-sec signal was also recorded as a timing trace. Fig. 6 shows (left to right) a cathode-ray oscilloscope with external sweep generator, relay racks containing one five-channel differential transformer driving unit (middle of left rack) and eight strain indicators with power supplies, and the string oscillograph (at far right). This constituted all the auxiliary equipment. Errors in measurement were estimated as shown in Table 3. The consistency of results indicated that some errors were compensating so as collectively to be on the order of 5%.

TABLE 3.—EXPERIMENTAL ERRORS, IN PERCENTAGES

Measurement	Systematic			Accidental	
	Gage factor or calibration device	Nonlinearity		Scaled Amplitude	
		Gage	Galvanometer	Response	Calibration
Strain	2	...	1	2	2
Displacement	0.5	6	1	3	3
Load	0.5	...	1	2	2

EXPERIMENTAL RESULTS AND COMPARISON WITH THEORY

Free Vibration.—Vibration tests were conducted using springs of two different stiffnesses. The frequencies of vibration corresponding to the states of resonance are given in Table 4, which also list the frequencies as computed from Eq. 13. (See Appendix II, Numerical Example.)

For all the flexural modes up to the seventh (that is, for $i > 1$) the agreement between the observed and computed values is excellent. Such a correlation with the use of two different values of elastic support was interpreted as reassuring evidence of the validity of the theory and also of the satisfactory mechanical arrangement of the supported ring. This was one of the most important results of the experiment.

TABLE 4.—FREQUENCIES OF FREE VIBRATION ON AN ELASTICALLY SUPPORTED RING, IN CYCLES PER SECOND

Mode, i	Modulus of Elastic Support, k (Lb per In. per In.)			
	6.8		31.6	
	Computed[a]	Observed	Computed[a]	Observed
1 (Eq. 13)	28.74	37.5	62.0	73.8
1(a) (Eq. 32)	33.50	37.5	72.1	73.8
2	41.49	42.6	80.9	84.8
3	68.50	66.0	100.6	98.9
4	115.5	111.3	138.0	134.7
5	180.0	176.9	195.2	191.9
6	260.7	254.0	271.8	276.5
7	356.8	358.0	365.5	362.0
8	468.2	...	475.0	474.0

[a] Computations, except for $i = 1$(a), were based on Eq. 13, with $m = 0$ $\gamma = 0.322$ lb per in.^{3} and $E = 29.2 \times 10^6$ lb per in^2.

Rigid-Body Translation and the Presence of Tangential Constraint.—One obvious discrepancy in frequencies is noted in the case of rigid-body translation where $i = 1$ (see Table 4). An alternate calculation of rigid-body translation of a radially supported ring was made by the following elementary procedure, with the assumption that all springs were combined as one

of the constant K.

$$K = \sum k_n \cos \theta_n = 4\, k_n\, r \int_0^{\frac{\pi}{2}} \cos \theta \, d\theta = 4\, k_n\, r \quad \ldots\ldots\ldots (31)$$

and

$$p'_1 = \sqrt{\frac{K\, g}{2\, \pi\, r\, A\, \gamma}} = \sqrt{\frac{2}{\pi}} \sqrt{\frac{k_n\, g}{A\, \gamma}} \quad \ldots\ldots\ldots\ldots\ldots (32)$$

in which p'_1 is the circular frequency of rigid-body motion, and k_n is the stiffness constant associated with each individual spring that furnishes radial support.

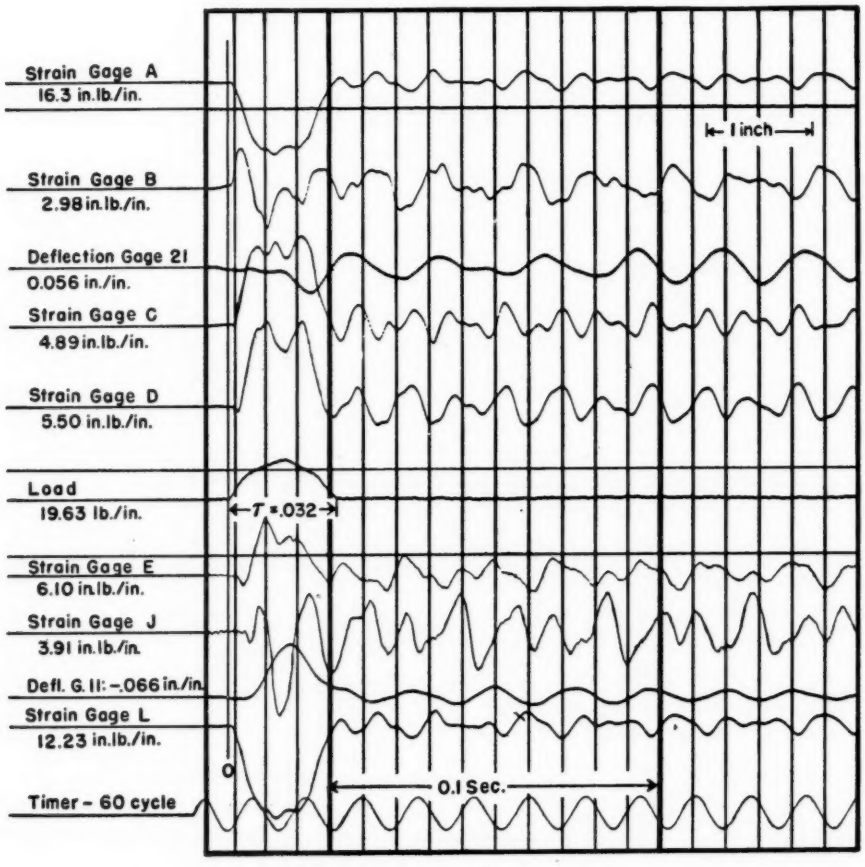


FIG. 7.—OSCILLOGRAM OF RING RESPONSE

This derivation of rigid-body motion takes into account the capacity of a radial spring to exert tangential constraint. As shown in Table 4, the frequency of vibration p'_1 as obtained with this computation is in better agreement with the observed frequency than with the value computed using Eq. 13. In this application of the theoretical analysis it was assumed that only radial elastic sup-

port could occur with radial displacement, but it is believed that, during the test the coil springs, intended only for radial support, inadvertently offered tangential support. Thus the condition that $m = 0$ was not experimentally achieved, and the difference between computed and observed frequencies for the case of rigid-body motion is explained.

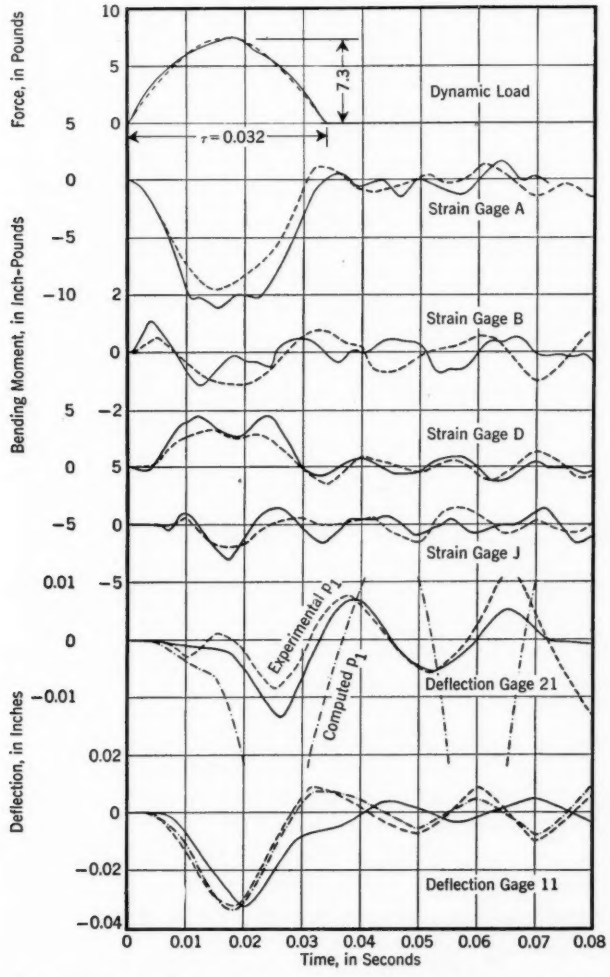


FIG. 8.—COMPARISON OF EXPERIMENTAL AND THEORETICAL RESULTS, $k = 6.8$

The frequency corresponding to rigid-body motion was of some significance in computing radial displacements but not in computing the bending moments because the term $i = 1$ does not appear in the moment equation.

Results of Tests with Transient Loading.—Transient loading of the ring was produced by dropping the pendulum of selected mass through a predetermined

stroke. Simultaneously recorded were the indications of the load, strain at seven stations, deflections at two stations, and a 60-cycle-per-sec timing pulse. Scales were established for these various quantities by using the height of calibration steps recorded prior to each dynamic test. A typical oscillogram is shown in Fig. 7, corresponding to a test in which k was equal to 6.8 lb. per in. per in. and the force was applied directly, using a 2.21-lb pendulum having a

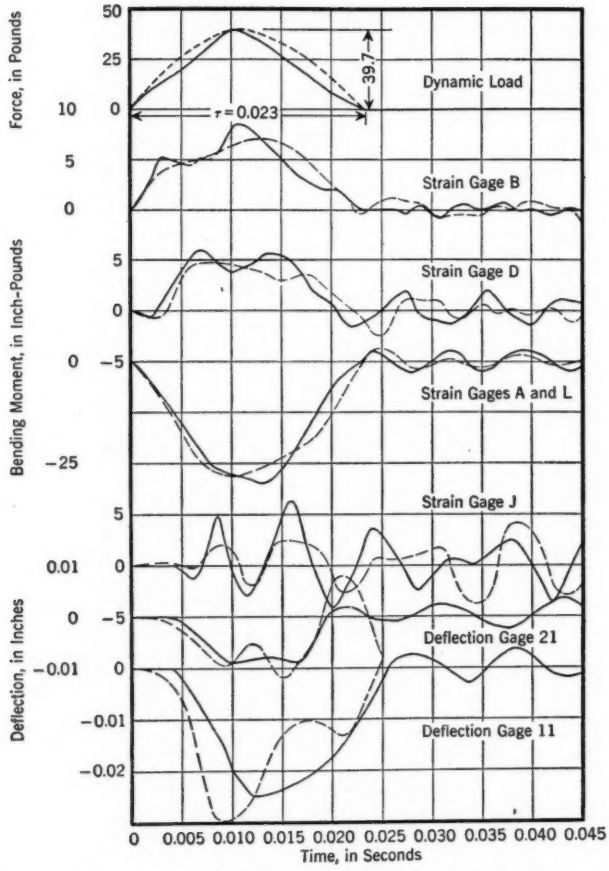


FIG. 9.—COMPARISON OF EXPERIMENTAL AND THEORETICAL RESULTS, $k = 31.6$

6-in. stroke. Along the left margin, calibration scales are given for each gage. The variety in character of response at different stations is immediately apparent. At stations near the point of application of load (stations A and L), the bending moments during loading are similar in pattern to the loading pulse, whereas irregular undulations appear elsewhere. The delay in incidence of response at stations remote from the point of impact is also apparent.

Displacements Produced by Transient Loading.—The computed theoretical responses were plotted separately in Figs. 10 and 11. (For computations and

plotted curves, see Appendix II.) Aggregate values of the responses corresponding to $k = 6.8$ were plotted as broken lines in Fig. 8. Experimental results in the form of oscillogram records were plotted as solid lines for comparison. This was also done for the value of $k = 31.6$, as shown in Fig. 9.

It is apparent that agreement between the computed and observed behavior is found both during and after the transient disturbance. Not only are the intensities of response comparable, but similar patterns are found in both the experimental and computed curves. The extent of agreement both confirms the theory and demonstrates that experimental techniques were satisfactory.

Summary and Conclusions

Restatement of Important Equations of Response.—Equations have been derived for radial displacements and bending moments in a circular ring having elastic radial and tangential support and subjected to either steady-state forced vibration or transient loading. These equations are in the form of infinite series given by Eqs. 23 for steady-state forced vibration and for the period of action of a half-sine pulse, and by Eqs. 28 for free vibration subsequent to a transient pulse. Solutions for the limiting case of static loading are given by Eqs. 29.

The frequency of free vibration, including the effects of radial and tangential elastic support, is given by Eq. 13. It was found that the addition of moderate elastic tangential support influences primarily the radial displacements and the frequency of rigid-body translation.

Experimental Investigation.—An experiment has been performed with a ring of 24-in. diameter having a cross section 1 in. by 0.125 in. supported radially with thirty-eight coil springs and subjected to a radial impulse having as its time history a half-sine pulse. The intensity and duration of the pulse and the stiffness of the springs were varied, and the response was observed.

A special pulse generator and a dynamometer were developed for the purpose of applying an analytically described transient load to the ring, and for measuring this load.

Measurements were made of the frequencies of free vibration and of the bending moments and radial displacements produced by transient loading. The experimental errors present in the test were carefully investigated, and (on the basis of consistency of results) were of the order of 5%.

Agreement Between Theory and Experiment.—It was observed that, with steady-state forced vibrations, the frequencies of the various modes agreed with values computed by the theory within a small percentage, even up to the seventh flexural mode. Furthermore, the computed curves of displacement and bending moment as functions of time agreed reasonably well with experimental data.

Conclusions.—It is therefore concluded that (1) the experimental procedures in this test were satisfactory, (2) the simplifying assumptions of the analysis do not vitiate its accuracy, and (3) the mathematical analysis for an elastically supported ring is applicable to a ring in which the thickness is small in comparison to the diameter.

Acknowledgments

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APPENDIX I. NOMENCLATURE

The following letter symbols, adopted for use in the paper and its discussions, conform essentially to American Standard Letter Symbols for Structural Analysis (ASA-Z10.8–1942) and to American Standard Letter Symbols for Mechanics of Solid Bodies (ASA-Z10.3–1948) prepared by a committee of the American Standards Association with ASCE representation, and approved by the Association in 1942 and 1948, respectively.

A = area of a cross section of the ring;
a_i = generalized coordinates;
B = normal force on a cross section of the ring;
C, D = constants;
d = stroke of the pendulum;
E = modulus of elasticity;
F_i, H_i = constants;
g = acceleration of gravity;
I = moment of inertia about the x-axis of a cross section of the ring;
i = integer designating the mode of vibration;
K = constant for combined springs of constant k_n which simulate the elastic foundation;
k = modulus of radial elastic foundation;
k_n = stiffness constant for individual springs that simulate elastic foundation;
L = length of the pendulum;
M = flexural couple acting about the x-axis on a cross section of the ring —positive if it increases the curvature of the ring;
m = modulus of tangential elastic foundation;
N = shear in a radial direction on a cross section of a ring;
P = external load (force per inch of circumference)—positive toward the center of the ring;
P_o = maximum value of P;
p = circular frequency of a vibrating ring;
p_i = circular frequency of a vibrating ring at mode i;
Q_i = generalized external force, measured positive toward the center;

r = radius of the ring;
s = circumference of the ring;
T = kinetic energy;
t = time variable;
t_1 = particular time at which behavior is stated;
U = characteristic radial displacement function;
u = radial displacement, measured positive toward the center of the ring;
V = potential energy;
v = terminal velocity of the body at the time of impact with the ring;
W = tangential displacement function;
W_p = weight of body striking ring;
W_r = combined weight of the ring and spring system;
w = tangential displacement of the ring, measured positive if counterclockwise;
α, β = constants;
Γ = time function;
γ = density;
$\eta_i = \omega/p_i$;
θ = central angle of the ring, measured positive counterclockwise;
ν, μ = constants defined by Eqs. 9*a* and 9*b*;
τ = duration of the half-sine pulse;
ω = circular frequency of the external periodic exciting force $\omega = \pi/\tau$; and
δ = constant.

APPENDIX II. NUMERICAL EXAMPLE OF SERIES SOLUTION WITH DYNAMIC LOADING

To evaluate the analysis, computations have been made of an example of the behavior of a radially supported ring with impact loading.

For these calculations, the parameters were assigned the values of those of the test ring: $P = 1$ lb, $r = 12$ in., $A = 0.125$ in.^{2}, $I = 162.8 \times 10^{-6}$ in.^{4}, $E = 29.2 \times 10^6$ lb per in.^{2}, $\gamma = 0.322$ lb per in.^{3}, $k = 6.8$ and 31.2 lb per in. per in., $m = 0$, and $\tau = 0.032$ sec and 0.23 sec.

It will be noted that the value chosen for density γ includes a correction for effective mass of the springs, a part of each of which may be expected to vibrate with the ring. This correction was applied as follows: The density of the cold-rolled steel was 0.292 lb per in.^{3} and the weight of the ring was 2.73 lb. The springs weighed 0.45 lb, of which 0.33 lb was assumed active. Thus, the effective density was increased by 11% to $\gamma = 0.322$. The frequencies p_i for both rigid-body motion and flexural vibration were computed from Eq. 13 and were listed in Table 4. A graphical demonstration of the first six modes of vibration is presented in Fig. 10, in which the arrows indicate the point and direction of application of the pulsating load.

By employing the values of τ (the duration of the half-sine pulse) actually observed during experiments, the response of the ring has been computed for the time intervals during and after the pulse. With the elastic support of modulus $k = 6.8$, τ was 0.032 sec, and with $k = 31.6$, τ was 0.023 sec.

The bending moments were computed at positions on the ring corresponding to four stations A, B, D, and J, and displacements were computed at stations 11

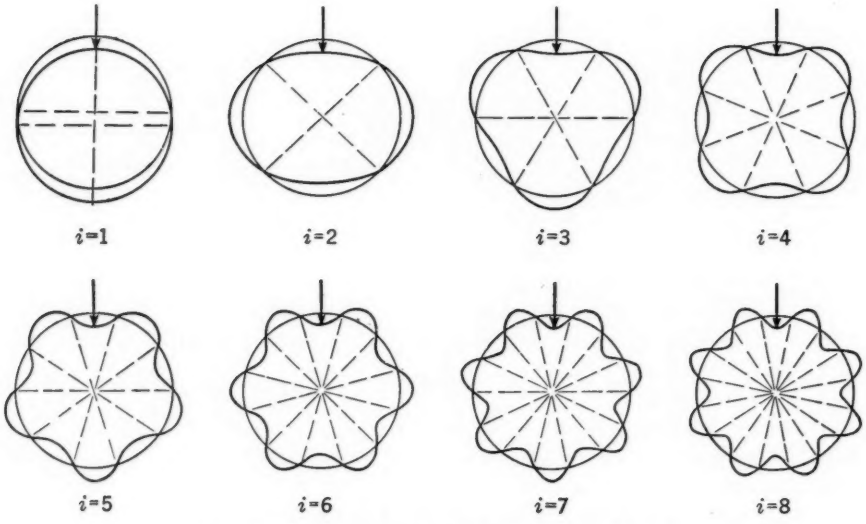

FIG. 10.—RING DEFORMATIONS FOR DIFFERENT MODES

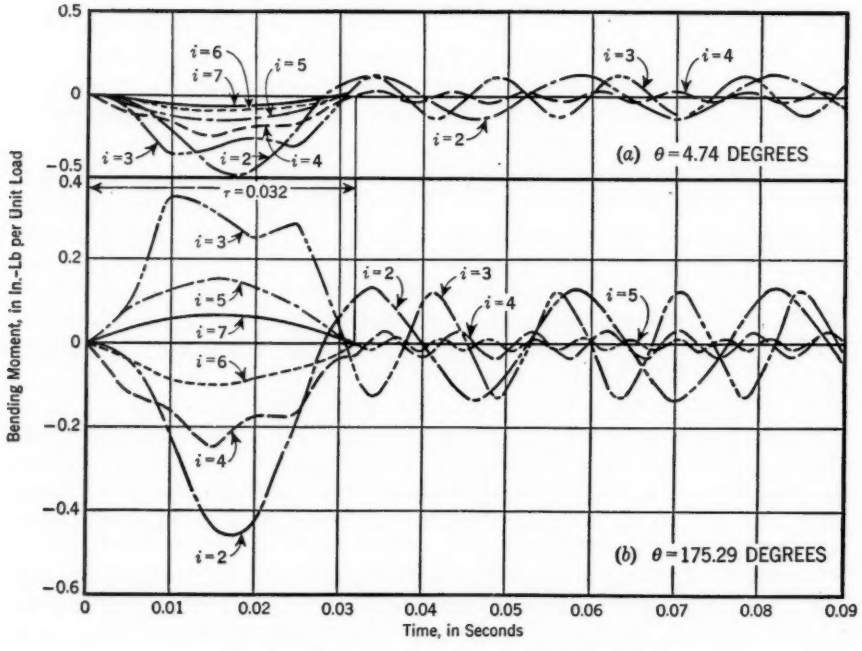

FIG. 11.—RESPONSE CURVES, $k = 6.8$ LB PER IN. PER IN.

and 21; also, experimental observations were made at all the stations whose locations were shown in Fig. 5. The computed response corresponding to contributions of each of the different modes up to $i = 7$ are plottedf or two values of θ in Figs. 11 and 12; none were shown where the amplitude was less than 2% of the maximum value. Because the observed frequency of rigid-body vibra-

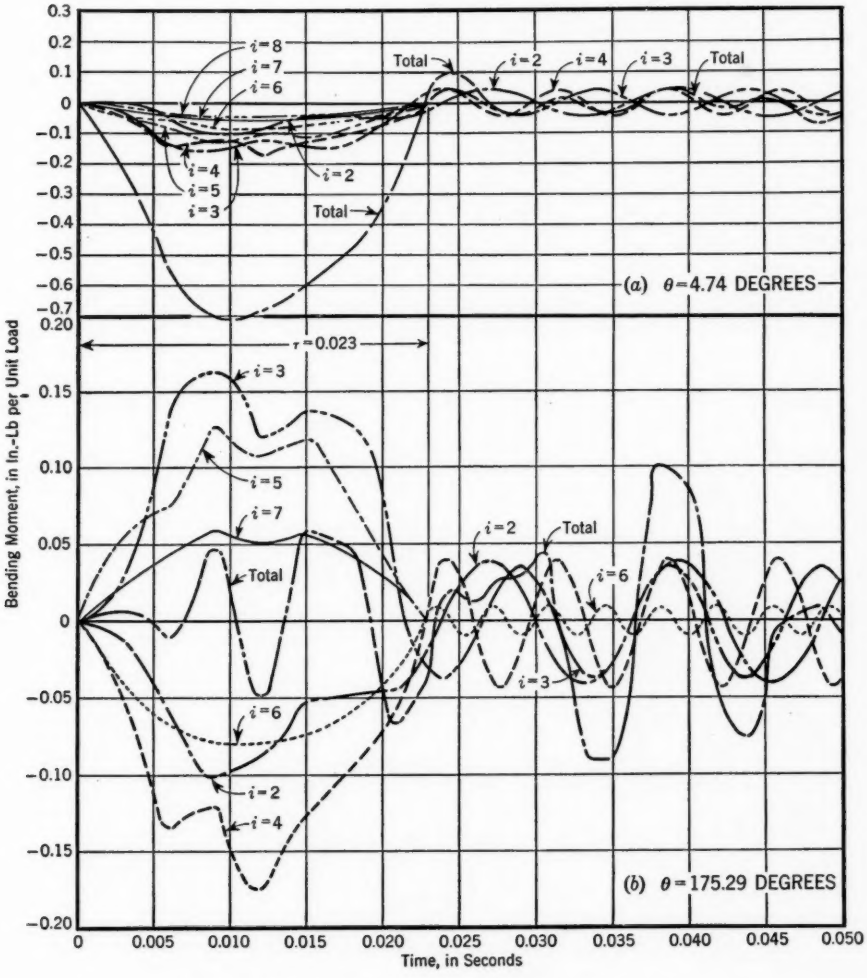

FIG. 12.—RESPONSE CURVES, $k = 31.6$ LB PER IN. PER IN.

tion was found to differ from the computed frequency as shown in Table 4, numerical calculations of response were made for $i = 1$, using observed values of p_1 as well as computed values.

The total response at any instant is the algebraic sum of the contributions from all modes and plots of the aggregate responses are given in Figs. 8 and 9, where experimental and theoretical results are compared.

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